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Rigorous Spectral Analysis of the Metal–Insulator Transition in a Limit-Periodic Potential

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We consider the limit-periodic Jacobi matrices associated with the real Julia sets of $f_\lambda(z) = z^2 - \lambda$ for which $\lambda \in [2, \infty)$ can be seen as the strength of the limit-periodic coefficients. The typical local spectral exponent of their spectral measures is shown to be a harmonic function in λ decreasing logarithmically from 1 to 0.

KEY WORDS: Julia matrices; local spectral exponents.

One of the possible mechanisms leading to a metal-insulator transition within a one-particle theory is based upon quantum effect in almost-periodic systems. Other than the Anderson transition in a disordered system, this transition is expected to be continuous: as the strength of the almost-periodic potential increases, the Hausdorff dimension of the singular continuous local density of states should decrease from 1, the Hausdorff dimension of an absolutely continuous measure, towards 0, the Hausdorff dimension of a pure-point measure.

The purpose of this note is to exhibit an example for which the above scheme can be proved to be correct. Using a result of Manning,[12] we hence complete the work of Bessis *et al.*[4] on the limit-periodic Jacobi matrices associated to the real Julia sets of $f_\lambda(z) = z^2 - \lambda$, $z \in \mathbf{C}$ and $\lambda \in [2, \infty)$. Further examples for which a proof of a similar result should be within reach are the Jacobi matrices associated to iterated function systems,[10] the open question being the proof of their almost periodicity.

The implications of the above spectral metal-insulator transition for the transport properties of the system are based on Guarneri's inequality.[7,9,1]

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It states that the diffusion exponents σ_q, $q > 0$, defined by the time asymptotic behavior of the moments of the position operator by $\langle \psi | \, |X(t)|^q \, |\psi \rangle \sim t^{q\sigma_q}$, are bounded below by the Hausdorff dimension of the local density of states, notably the spectral measure of ψ. For a more detailed characterization of the diffusion exponents, not only the Hausdorff dimension of the spectral measure intervenes, but also their multifractal properties.[11] However, we expect the diffusion exponents also to converge to zero in the limit $\lambda \to \infty$. We finally note that a diffusion exponent σ_2 different from 1 leads to anomalies in Drude's formula.[15]

We first recall from ref. 4 some facts about the Jacobi matrices H_λ associated to f_λ, $\lambda \in [2, \infty)$. Let $(|n\rangle)_{n \in \mathbf{N}}$ be a basis of the Hilbert space $\ell^2(\mathbf{N})$, then H_λ is given by

$$H_\lambda \, |n\rangle = t_{n+1} \, |n+1\rangle + t_n \, |n-1\rangle \tag{1}$$

where the sequence $(t_n)_{n \in \mathbf{N}}$ in $[0, \sqrt{\lambda}]$ is calculated from the recursion relations

$$(t_{2n})^2 + (t_{2n+1})^2 = \lambda, \qquad t_{2n-1} t_{2n} = t_n$$

with initial condition $t_0 = 0$. This sequence $(t_n)_{n \in \mathbf{N}}$ is limit-periodic. The spectrum of H_λ is the Julia set J_λ of f_λ, namely the set of all $z \in \mathbf{C}$ such that the sequence $(f_\lambda^k(z))_{k \in \mathbf{N}}$ is bounded.

Let μ_λ denote the spectral measure of H_λ associated to the state $|0\rangle$. Its Hausdorff dimension is defined by the infimum of the Hausdorff dimensions of all Borel subsets $S \subset \mathbf{R}$ satisfying $\mu_\lambda(S) = 1$.[16] According to Rogers' and Taylor's theorem,[14] it is also equal to the μ_λ-essential supremum of all spectral exponents $\alpha_{\mu_\lambda}(E)$ given by

$$\alpha_{\mu_\lambda}(E) = \sup \left\{ \gamma \in \mathbf{R} \;\middle|\; \int d\mu_\lambda(E') \, |E' - E|^{-\gamma} < \infty \right\}$$

For more details on the fractal analysis of the spectral measure μ_λ, we refer to refs. 9, 13, and 15. We have now introduced all the notions necessary in order to state the result.

Theorem. The Hausdorff dimension $\dim_{\mathrm{H}}(\mu_\lambda)$ of the spectral measure μ_λ is a harmonic function of $\lambda \in [2, \infty)$. For $\lambda = 2$, it is equal to 1, and in the limit $\lambda \to \infty$, one has $\dim_{\mathrm{H}}(\mu_\lambda) = \log(4)/(\log(\lambda) + o(1))$. Furthermore, the spectral exponents $\alpha_{\mu_\lambda}(E)$ are μ_λ-almost surely constant and equal to $\dim_{\mathrm{H}}(\mu_\lambda)$.

As shown in refs. 2 and 4, the spectral measure μ_λ is nothing but the invariant, ergodic, maximal entropy measure of the dynamical system

(J_λ, f_λ). The proof of the theorem is therefore a direct application of a result of A. Manning[12] linking the Hausdorff dimension of the measure μ_λ to the quotient of measure-theoretic entropy and Lyapunov exponents of the dynamical system (J_λ, f_λ) with respect to μ_λ. This is an example of an extension of Pesin's equality to singular measures. Let us remark that, on the level of the Jacobi matrices H_λ, the dynamics turns out to be an exact renormalization procedure.[4,2] More precisely, if D is the dilation operator by a factor 2 (that is, $D^* |n\rangle = |2n\rangle$), then one has $H_\lambda D = Df_\lambda(H_\lambda)$.

Recently, Jitomirskaya and Last have extended the Gilbert–Pearson theory of subordinacy in order to develop a tool for computing spectral exponents.[8] The above dynamical system approach is complementary and it may turn out to be more fruitful for the explicit calculation of the dimension of the spectrum also for other examples.

Note that the last statement in the theorem does not exclude that there are energies for which the spectral exponent is not equal to the Hausdorff dimension of the measure. For example, if E is equal to the band edge $(1+\sqrt{1+4\lambda})/2$ or any of its preimages (which form a dense subset of J_λ), the spectral exponent was calculated explicitly by Bessis *et al.*[3] and is equal to

$$\alpha_{\mu_\lambda}(E) = \frac{\log(2)}{\log(1+\sqrt{1+4\lambda})} \tag{2}$$

For small λ, these spectral exponents are considerably smaller than the typical one (compare Fig. 1) and are due to van Hove singularities. For example, in the case $\lambda = 2$, the spectral measure is absolutely continuous on $[-2, 2]$ with density $1/(\pi\sqrt{4-E^2})$ which has singularities at the band edges with exponent 1/2.

We conclude by an outline of the proof. The key point is Manning's volume lemma[12] which states

$$\dim_H(\mu_\lambda) = \frac{H_{\mu_\lambda}(f_\lambda)}{\chi_\lambda} \tag{3}$$

where $h_{\mu_\lambda}(f_\lambda)$ is the dynamical entropy of f_λ and $\chi_\lambda = \int d\mu_\lambda(E) \log(|2E|)$ its Lyapunov exponent (both with respect to μ_λ). Using symbolic dynamics,[5] one verifies that $h_{\mu_\lambda}(f_\lambda) = \log(2)$. Now recall Brolin's weak limit representation of the measure μ_λ:[5]

$$\int d\mu_\lambda(E)\, g(E) = \lim_{k\to\infty} \frac{1}{2^n} \sum_{E \in f_\lambda^{-k}(z_0)} g(E), \qquad g \in C_0(\mathbf{R})$$

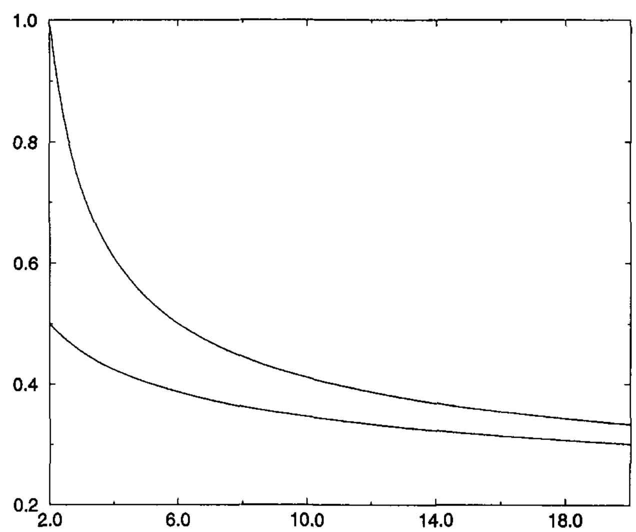

Fig. 1. Upper curve: Hausdorff dimension of the spectral measure μ_λ as a function of λ (calculated numerically using a weighted Bernoulli process choosing the branches of the inverse map of f_λ); lower curve: spectral exponent at the band edge and its preimages as given in (2).

which is independent of $z_0 \in \mathbf{C}$ excluding one exceptional point. Hence the Lyapunov exponent is given by

$$\chi_\lambda = \log(2) + \lim_{k \to \infty} \frac{1}{2k} \log \left(\prod_{E \in f_\lambda^{-k}(z_0)} |E| \right)$$

Now, by Brolin's calculation,[5]

$$\prod_{E \in f_\lambda^{-k}(z_0)} E = f_\lambda^k(0) - z_0$$

In fact, the roots of the polynomial $f_\lambda^k(z) - z_0$ are exactly the energies appearing in the above product. Thus the Lyapunov exponent is equal to $\log(2) + \log(|\phi_\lambda(0)|)$ with

$$\phi_\lambda(z) = \lim_{k \to \infty} (f_\lambda^k(z))^{1/2k} \tag{4}$$

where we have used that 0 is in the attraction basin of the superattracting fixed point ∞ so that z_0 can be suppressed. Now (4) is exactly the Böttcher representation of the conjugation of f_λ at ∞,[6] that is, it verifies $\phi_\lambda(z)^2 = \phi_\lambda(f_\lambda(z))$. It is straightforward to verify that $\phi_\lambda(0)$ is analytic in λ for $\Re e(\lambda) > 2$. Hence by (3) the Hausdorff dimension of μ_λ is harmonic in λ. Furthermore $\phi_\lambda(z) = z(1 + o(1))$ for z in a neighborhood of ∞. Therefore

$$\chi_\lambda = \log(2) + \tfrac{1}{2}\log(|\phi_\lambda(\lambda)|) = \tfrac{1}{2}\log(\lambda) + o(1)$$

which proves the next statement of the theorem.

Finally according to ref. 17, the spectral exponents can be calculated from the Green's function of the spectral measure μ_λ by means of the formula $\Im m\, G(E - \imath\varepsilon) \sim \varepsilon^{\alpha_{\mu_\lambda}(E)-1}$ as $\varepsilon \to 0$. G satisfies the functional equation $G(z) = zG(z^2 - \lambda)$ because for the resolvant of H_λ the renormalization equation $D1/(z - H_\lambda)\, D^* = z/(z^2 - \lambda - H_\lambda)$ holds.[2] From the functional equation it is now possible to verify that $\alpha_{\mu_\lambda}(E) = \alpha_{\mu_\lambda}(f_\lambda(E))$ for any $E \in \mathbf{R}$ and $\lambda > 2$. Because the measure μ_λ is ergodic and the mapping $E \mapsto \alpha_{\mu_\lambda}(E)$ is borelian,[15] it follows that the spectral exponents are μ_λ-almost surely constant.

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